NASA CONTRACTOR REPORT

NASA CR-958

NASA CR-958

STABILITY OF SHEAR FLOW WITH DENSITY GRADIENT AND VISCOSITY

by Richard L. Baker, Tzvi Rozenman, and Herbert Weinstein

Prepared by
ILLINOIS INSTITUTE OF TECHNOLOGY
Chicago, Ill.
for Lewis Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • FEBRUARY 1968

NASA CR-958

STABILITY OF SHEAR FLOW WITH DENSITY GRADIENT AND VISCOSITY

By Richard L. Baker, Tzvi Rozenman, and Herbert Weinstein

Distribution of this report is provided in the interest of information exchange. Responsibility for the contents resides in the author or organization that prepared it.

Prepared under Grant No. NsG-694 by
ILLINOIS INSTITUTE OF TECHNOLOGY
Chicago, Ill.

for Lewis Research Center

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – CFSTI price $3.00

FOREWORD

Research related to advanced nuclear rocket propulsion is described herein. This work was performed under NASA Grant NsG-694 with Mr. Maynard F. Taylor, Nuclear Systems Division, NASA Lewis Research Center as Technical Manager.

ABSTRACT

The stability of the mixing region between co-flowing streams of different velocity and density has never been adequately investigated. The reason for this is that a general similarity solution for velocity and density profiles has not been available until recently.

In this work, the solution method of Lessen (1948) for the homogeneous case was extended to the heterogeneous case in an attempt to find a neutral stability curve for the more complex case. The extension was based on the recent similarity solution obtained by the authors.

A branch line of the neutral stability curve was found but curves with non-zero amplification and damping factors fell on the same side of the neutral stability curves.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

LIST OF SYMBOLS

Symbol	Definition
b	Width of mixing region for velocity
c	Constant of proportionality, i.e., $b = cx$
c_i	Amplification or damping factor
c_r	Velocity of propagation of disturbance
C	Molar density
$\mathcal{D}$	Diffusivity
d	Width of mixing region for velocity
f	Dependent variable for laminar similarity solution
F	Dependent variable for turbulent similarity solution
g	Acceleration of gravity
J	Richardson number
J_i	Molar flux of component i
K	$1/Sc$
K'	$1/Sc^{(t)}$
l	Prandtl's mixing length
P	Pressure or power input to sensor
p	Fluctuating power
R	Reynolds number
t	Time
u	x-component of velocity
U	Reference velocity
v	y-component of velocity
w	Dependent variable = $\int f d\eta$
x	Rectangular coordinate

Symbol	Definition
y	Rectangular coordinate
α	Wave number
$1/\beta$	Reference length for density
ϵ	Eddy viscosity
ξ	Independent variable in turbulent similarity solution
η	Independent variable in laminar similarity solution
λ	Constant related to velocity ratio, i.e., $\lambda = (u_1 - u_2)/(u_1 + u_2)$
Γ	Constant related to density ratio, i.e., $\Gamma = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$
X_1	Empirical constant
φ	Amplitude function
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
τ_{xy}	Apparent turbulent shearing stress
Nu	Nusselt number
Pr	Prandtl number
Ri	Richardson number
Sc	Schmidt number

Subscripts

A	Component A
B	Component B
i	Component i
1	Refers to lighter fluid

Symbol	Definition
2	Refers to heavier component
x	x-component
y	y-component

Superscripts

$-$	Time average
$*$	Molar average
$'$	Fluctuating component
(t)	Turbulent

I. INTRODUCTION

The hydrodynamic stability of the mixing region between parallel streams is of fundamental interest. Lord Rayleigh first discussed this stability problem as early as 1880. The problem is important because the width of the mixing region as well as the transport rates differ drastically between the cases of laminar and turbulent flow.

The cases considered here are the homogeneous - both streams of the same density, and the heterogeneous - the two streams have differing densities.

The homogeneous case has been discussed extensively in the literature because a similarity solution has been available to provide complete descriptions of velocity profiles in the mixing region. For this case, viscosity is the only stabilizing factor since in the inviscid limit the flow is inherently unstable.

The literature studies of the heterogeneous case have been limited because no similarity solution for this case has been available. The authors, however, in a preceding report (67) have obtained a general solution for the velocity and density profiles in the mixing region for this case. For the heterogeneous case, stabilizing factors in addition to viscosity are present. The criteria for stability, however, have not been clearly defined.

The purpose of this report is primarily to use the similarity solution (67) in an extension of a method reported by Lessen, (4) for the homogeneous case to attempt to obtain some stability criterion for the heterogeneous case.

II. BACKGROUND

II- 1 Homogeneous Case

The purpose of hydrodynamic stability studies is to provide criteria on the stability of particular flows. As previously mentioned, the viscosity is the only stabilizing effect for homogeneous flows and the stability criterion is given in terms of a critical Reynolds number.

Hydrodynamic stability studies may be classified as inviscid or viscous. An inviscid analysis can answer the question of whether a given laminar flow velocity distribution is basically stable or unstable. However, the effect of viscosity on disturbances in the flow is not considered. Thus an inviscid analysis cannot provide a critical Reynolds number.

Lord Rayleigh,[2] before the turn of the century, established two important theorems concerning the stability of homogeneous inviscid laminar velocity profiles based on the linearized theory of infinitesimal disturbances. The first of these states that a point of inflexion in the velocity profile constitutes a necessary condition for the occurence of instability. The second theorem states that the velocity of propagation of neutral disturbances is smaller than the maximum velocity of the mean flow.

Tollmien[15] later showed that the existence of a point of inflexion in the velocity profile constitutes a necessary and sufficient condition for the occurence of instability. However, Tollmien's proof is valid only for those flows which exhibit a vorticity maximum. While these theorems are for an inviscid fluid, they none the less provide information of fundamental interest in the study of hydrodynamic stability.

The problem of the hydrodynamic stability of a homogeneous half-jet flow has received considerable attention. Investigations of this problem may be differentiated in the following way. Lessen,[3] Lin[7] and Chiarulli[6] in their analyses solved the equations of continuity and momentum to obtain the velocity distributions in the mixing region. All other investigators assumed some form of an analytic function to represent the velocity distribution in the mixing region. It is well established that the velocity profile in the mixing region of a half jet contains a point of inflexion. Thus, according to Rayleigh's first theorem the flow is unstable in the inviscid case.

In hydrodynamic stability analyses, the flow in the mixing region is first assumed to be stable. The equations of continuity and motion are solved to obtain the laminar velocity profile in the mixing region. This profile is then investigated to determine if it is stable or unstable.

Lessen[3] obtained a similar solution for the velocity distribution in the mixing region of a half-jet flow by first developing an asymptotic expression which represented the solution for large negative values of the independent similarity variable $\eta = y\sqrt{\frac{U}{\nu x}}$. In this expression, x and y are rectangular coordinates, ν is the kinematic viscosity and U is a reference velocity. He then used the method of analytic continuation (Taylor series) to numerically integrate his differential equation and obtain the solution. He next employed the same method to obtain a solution of the Orr-Sommerfeld equation of hydrodynamic stability. (See section III-2). However, Lessen was unable to obtain a complete neutral curve because of inaccuracies in his solution method at low values of the Reynolds number. He thus was not able to give a value of the critical Reynolds number for stable flow. The partial neutral curve that he obtained indicated that the flow is unstable except for very low Reynolds numbers.

Lessen and Ko[16] have very recently extended Lessen's earlier work. By employing a slightly different solution method they were able to obtain a complete neutral curve and a value of the critical Reynolds number of 3.6. However, they showed that if the solution is corrected for the non-parallelism in the flow that exists at low Reynolds numbers; then the minimum critical Reynolds number becomes 12.

The curve of neutral stability for a homogeneous flow is given in a wave number - Reynolds number plane.

For very large values of the Reynolds number the curve of neutral stability approaches an asymptote. The asymptotic value of the wave number for infinite Reynolds number is known as the cut-off wave number. The flow is stable for disturbances with wave numbers greater than the cut-off wave number for all values of the Reynolds number. Lessen obtained a value of the cut-off wave number of 0.395.

Chiarulli[6] applied Heisenbergs[17] solution method in his analysis of the stability of half-jet type flows. He used Goertler's[4] numerical method to solve the continuity and momentum equations to obtain the velocity profiles in the mixing region. A complex integral expression was derived from which the eigenvalues were to be obtained for the curve of neutral stability. The inviscid case (infinite Reynolds number) was first considered. A value of the cut-off wave number of 0.51 was obtained as opposed to Lessen's value of 0.395. The complexity of the expressions involved made it impossible to obtain a solution for the viscous case and thus a critical Reynolds number.

Lin[7] later extended Chiarulli's analysis to the case of compressible flow. He concluded that when the relative speed of the two parallel streams exceeds the sum of their velocities of sound, subsonic oscillations can not occur. He also showed that a necessary condition for the possible occurence of subsonic disturbances is that somewhere in the flow field

$$\frac{d}{dy}\left(\rho \frac{dw}{dy}\right) = 0 \qquad \text{II-1.1}$$

where y is the transverse coordinate, ρ is the density and w is a dimensionless velocity distribution. The above conclusions are from an inviscid analysis. A recent study of the inviscid stability of a compressible half-jet type flow has been given by Lessen, et. al.[18]

An analytic function or a broken line was used to approximate the velocity distribution in the mixing region of a half-jet in the stability analyses of Betchov and Szewczyk,[19] Curle[20] and Esch.[21] Curle used the function

$$U = V \tanh (y/L)$$

to represent the velocity distribution in the mixing region. He obtained a minimum critical Reynolds number, $R = VL/\nu$ of 8.9.

Betchov and Szewcyk also used the function

$$U = U_o \tanh (y/L)$$

to represent the velocity distribution in the mixing region. No minimum critical Reynolds number was found. However, a spreading layer in which L in the above equation increases with time according to the relationship

$$L = \sqrt{4\nu t}$$

was also considered. Using their results for L equal to a constant and applying some physical reasoning based on the total amplification available to small perturbations, they predicted a critical Reynolds number of 150.

Esch[21] used a piece-wise linear profile in his analysis. Unstable disturbances were found at all values of the Reynolds number. This problem has also been considered by Tatsumi and Gotoh[22] and Carrier.[23]

II -2 Heterogeneous Case

The instability of a stratified heterogeneous fluid when the different layers are in relative motion is classically referred to as Kelvin-Helmholtz instability. Consideration of this problem has indicated that stabilizing influences in addition to the viscosity affect the stability of the flow. First there is the effect of what has been called the "heterogeneity of inertia". This arises from the fact that the resistance per unit volume to accelerating forces is not constant because of the density variation. If the flow takes place in a gravitational field, an additional effect is apparent. Work must be done to effect the interchange of volume elements against the gravitational field. The performance of this work decreases the net kinetic energy available for transfer from the mean flow to the fluctuating components.

The effect of the heterogeneity of inertia has been studied by Menkes.[24] He considered the flow to be in the absence of a gravitational field. He represented the velocity distribution in the mixing region by

$$U(y) = \tanh(y/d)$$

and assumed that the density decreased exponentially with height. He demonstrated that disturbances with wave numbers larger than the width of the mixing region are stable and that a necessary condition for instability is that

$$d\left(\rho \frac{du}{dy}\right)$$

should change sign somewhere in the flow field. It is interesting to note that this is the same condition as that given by Lin[7] for the inviscid homogeneous compressible flow case. Menkes analysis is also for the inviscid case. The stability criterion for viscous flow including the effect of the heterogeneity of inertia is still given in terms of a critical Reynolds number.

The effect of a gravitational field was first discussed by Richardson[25] and by Prandtl.[26] Richardson's discussion concerned the supply of energy to and from atmospheric eddies. Richardson derived a stratification parameter which determined if the kinetic energy associated with velocity fluctuations would increase. This parameter subsequently became known as the Richardson number. Prandtl pointed out that the stability of viscous density stratified flows depends on the Richardson number as well as the Reynolds number. The gradient form

Richardson number is given by

$$Ri = \frac{\frac{-g}{\rho}\left(\frac{\partial\rho}{\partial y}\right)}{\left(\frac{\partial u}{\partial y}\right)^2} \qquad \text{II-2.1}$$

where g is the acceleration of gravity, ρ is the density, u is the velocity and y is the coordinate in the direction of the gravity field. Defining d as the reference length for velocity, $1/\beta$ as the reference length for density and referring all velocities to the reference velocity V; the Richardson number may be defined as

$$J = \frac{g\beta d^2}{V^2} \qquad \text{II-2.2}$$

The stability of density stratified flows in the presence of a gravity field has been considered by Taylor,[27] Goldstein,[28], Drazin,[29] Miles[30] and others.[31,32] These analyses were all for the inviscid case and did not include the effect of the heterogeneity of inertia.

Taylor considered several problems of three of four superposed streams. The velocity in each layer was constant or varied linearly and the density was either constant or decreased exponentially with height. He concluded that there might be stability for

$$J > 1/4$$

in the limiting case of a continuous density distribution.

Goldstein considered a three layer flow. The velocity was constant in the upper and lower layers and varied linearly in the middle region. The density decreased exponentially with height in the upper two layers and was constant in the lower layer. He concluded that disturbances can be neutrally stable only if

$$J \leq 1/4$$

and therefore the flow is stable for

$$J > 1/4$$

and unstable for

$$J < 1/4$$

Drazin[29] considered the stability of a shear layer between parallel streams with density stratification. The velocity distribution in the mixing region between the two streams was represented by

$$U = V \tanh (y/d)$$

and the density was assumed to decrease with height according to the relation

$$\rho = \rho_o \, e^{-\beta y}$$

The gravity field was taken to be perpendicular to the main flow. Drazin obtained a curve of neutral stability in a wave number - Richardson number plane and concluded from this that the critical Richardson number is 1/4, i.e., the flow is stable for

$$J > 1/4$$

and unstable for

$$J < 1/4$$

Miles[30] used a hyperbolic tangent function to represent both the velocity and the density distribution in the shear layer. The slower moving stream corresponded to the higher density and the gravitational field was again assumed to be perpendicular to the main flow. He also obtained a neutral stability curve in the wave number - Richardson number plane and reached the same conclusions as Drazin, i.e., stable flow for

$$J > 1/4$$

and unstable flow for

$$J < 1/4$$

The reoccurence of the value of 1/4 for the critical Richardson number in these analyses has not been satisfactorily explained. The following discussion is given in Chandrasekhar.[33]

If two neighboring volumes in a stratified flow are interchanged, the work that must be done against the acceleration of gravity, per unit volume, is given by

$$\delta W = -g\delta\rho\delta z$$

The kinetic energy which is available to do this work (per unit volume) is given by

$$1/2\rho[U^2+(U+\delta U)^2 - 1/2(U+U+\delta U)^2] = 1/4\rho(\delta U)^2$$

If the amount of work that must be done exceeds the amount of available kinetic energy, the displacement will not occur. Thus a sufficient condition for stability is

$$1/4\ \rho(\delta U)^2 < -g\delta\rho\delta z$$

or equivalently

$$\left(\frac{\partial u}{\partial z}\right)^2 < -4\ \frac{g}{\rho}\ \left(\frac{\partial \rho}{\partial z}\right)$$

$$Ri = -\frac{g\ \left(\frac{\partial \rho}{\partial z}\right)}{\rho\ \left(\frac{\partial u}{\partial z}\right)^2} > 1/4 \quad \text{(for stability)}$$

It is apparent that a necessary condition for instability is

$$Ri < 1/4$$

In the preceeding analysis, the effect of the density change $\delta\rho$ has been neglected in the expression for the available kinetic energy. This is equivalent to neglecting the effect of the heterogeneity of inertia and considering only the effect of the gravitational field. Thus, the conclusion reached in this manner agrees with that obtained for the same case by more involved analysis.

Menkes[34] in a later paper considered the stability of a density stratified shear layer including both the effect of the heterogeneity of inertia and the effect of the gravity field. He again assumed the velocity distribution to be given by a hyperbolic tangent function and that the density decreased exponentially with height. From his inviscid analysis he concluded that the value of the critical Richardson number depends on the magnitude of the non-dimensional density gradient. He obtained a family of neutral stability curves with the value of the critical Richardson number increasing as the non-dimensional density gradient decreases.

Two neighboring volumes in a stratified flow may again be considered to be interchanged. If the effect of the density change $\delta\rho$ is included in the expression representing the available kinetic energy, it can be demonstrated that the value of the critical Richardson number is decreased with an increase in the dimensionless density gradient, i.e.,

$$1/4\ \rho(\delta U)^2 + 1/2\ \delta\rho\delta UU < -g\delta\rho\delta z$$

or

$$\frac{-g\left(\frac{\partial\rho}{\partial z}\right)}{\rho\left(\frac{\partial u}{\partial z}\right)^2} - 1/2\ \frac{\frac{1}{\rho}\frac{\partial\rho}{\partial z}}{\frac{1}{u}\frac{\partial u}{\partial z}} > 1/4 \text{ (for stability)}$$

Thus, the conclusion reached in this manner agrees qualitatively with the analysis of Menkes.

Very few analyses of density stratified flows have been given that include the effect of viscosity. Shen[35] investigated the stability of laminar boundary layers with injection of a foreign gas. Although his analysis included viscous effects, he was unable to determine stability criteria for the viscous case. From his inviscid solution he concluded that injection of a heavier gas would result in a more stable laminar boundary layer than injection of a lighter gas.

Schlichting[36] considered the stability of a laminar boundary layer on a flat plate with a density gradient in the boundary layer and constant density outside of it. He found that the critical Reynolds number increased rapidly as the Richardson number increased. The critical Reynolds number for a Richardson number of zero (homogenerous flow) was found to be 575, while for a Richardson number of 1/24 the critical Reynolds number became infinite.

Clearly there is a definite need for additional analysis of the problem of the stability of half-jet type flows with density stratification. No analysis to date has been based on an acceptable similarity solution of the equations of continuity, momentum and diffusion. And at the present time no one has included the effect of viscosity in their analysis.

III Hydrodynamic Stability Analysis

III -1 The Disturbance Differential Equations

The Navier-Stokes equations for incompressible flow may be written

x-direction $$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = -\frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad \text{III-1.1}$$

y-direction $$\rho\left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right] = -\frac{\partial P}{\partial y} - \rho g + \frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right) \qquad \text{III-1.2}$$

If molecular diffusion is neglected, the equations of continuity and diffusion may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-1.3}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0 \qquad \text{III -1.4}$$

The most successful method of analyzing the stability of laminar flows is called "the method of small disturbances". The motion is decomposed into a mean flow and a disturbance superimposed upon it. The mean flow is regarded as steady and is described by the velocity components $\bar{u}$ and $\bar{v}$, the pressure $\bar{P}$ and the density $\bar{\rho}$. The corresponding quantities for the non-steady disturbance are denoted by u', v', p, and ρ' respectively. Thus, the instantaneous values of the velocity components, the pressure, and the density are given by

$$
\begin{aligned}
u &= \bar{u} + u' \\
v &= \bar{v} + v \\
P &= \bar{P} + p \\
\rho &= \bar{\rho} + \rho'
\end{aligned}
\qquad \text{III-1.5}
$$

Before proceeding, an additional simplifying assumption will be made. The mean velocity $\bar{u}$ and the average density $\bar{\rho}$ will be assumed to be a function of y only and the transverse velocity component will be assumed to be zero, i.e.,

$$
\begin{aligned}
\bar{u} &= \bar{u}\ (y) \\
\bar{\rho} &= \bar{\rho}\ (y) \\
\bar{v} &= 0
\end{aligned}
\qquad \text{III-1.6}
$$

Such a flow is described as a "parallel flow". For the two-dimensional mixing problem considered here, equations III-1.6 are a good approximation at reasonable distances downstream from the beginning of the mixing region, i.e., for larger values of the Reynolds number.

Introducing equations III-1.5 and III-1.6 into equation III-1.3, the continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \qquad \text{III-1.7}$$

Similarly, neglecting quadratic terms in the disturbance components, the diffusion equation becomes

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial \rho'}{\partial t} + \bar{u}\,\frac{\partial \rho'}{\partial x} + v'\,\frac{\partial \bar{\rho}}{\partial y} = 0 \qquad \text{III-1.8}$$

Since the mean flow is steady equation III -1.8 may be simplified to

$$\frac{\partial \rho'}{\partial t} + \bar{u}\,\frac{\partial \rho'}{\partial x} + v'\,\frac{\partial \bar{\rho}}{\partial y} = 0 \qquad \text{III-1.9}$$

After linearizing and subtracting the mean flow quantities, equations III-1.1 and III-1.2 become

$$\bar{\rho}\left[\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial u'}{\partial x} + v'\,\frac{\partial \bar{u}}{\partial y}\right] = -\,\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(\mu\,\frac{\partial u'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial u'}{\partial y}\right)$$

III-1.10

$$\bar{\rho}\left[\frac{\partial u'}{\partial t} + \bar{u}\,\frac{\partial v'}{\partial x}\right] = -\,\frac{\partial p}{\partial y} - \rho' g + \frac{\partial}{\partial x}\left(\mu\,\frac{\partial v'}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu\,\frac{\partial v'}{\partial y}\right)$$

III-1.11

The distubance has been assumed to be two dimensional on the basis of Yih's extension of Squire's theorem to the viscous heterogeneous case.[62]

The continuity equation, as given by equation III-1.7 may be integrated by introducing a stream function ψ' such that

$$u' = \frac{\partial \psi'}{\partial y}, \quad v' = -\,\frac{\partial \psi'}{\partial x} \qquad \text{III-1.12}$$

Equations III-1.12 and III-1.7 are based on the assumption that the effect of molecular diffusion may be neglected. The disturbance stream function ψ' is assumed to be represented by

$$\psi'(x,y,t) = \varphi(y)\ e^{i\alpha(x-ct)} \qquad \text{III-1.13}$$

where $\varphi(y)$ is the amplitude function of the fluctuation, α is the wave number of the disturbance and c is a complex quantity given by

$$c = c_r + i\ c_i \qquad \text{III-1.14}$$

In this equation c_r denotes the velocity of propagation of the wave in the x-direction and c_i determines the degree of damping or amplification depending on its sign.

From equations III-1.12 and III-1.13 it follows that

$$u' = -\frac{\partial\psi'}{\partial y} = \varphi'(y)\ e^{i\alpha(x-ct)}$$

$$v' = -\frac{\partial\psi'}{\partial x} = -\ i\alpha\ \varphi(y)\ e^{i\alpha(x-ct)} \qquad \text{III-1.15}$$

Introducing equations III -1.15 into equations III-1.10 and III -1.11 and eliminating pressure, the following ordinary fourth-order differential equation for the amplitude is obtained:

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi + \frac{\bar{\rho}'}{\bar{\rho}}[(\bar{u}-c)\varphi' - \bar{u}'\ \varphi] =$$

$$= \frac{g}{i\alpha\ \bar{\rho}}\ \frac{\partial\rho'}{\partial x}\ e^{-i\alpha(x-ct)} + \frac{\nu}{i\alpha}[\varphi^{IV} - 2\alpha^2\ \varphi'' + \alpha^4\varphi] \qquad \text{III-1.16}$$

The disturbance density is assumed to be represented by

$$\rho'(y) = R(y)\ e^{i\alpha(x-ct)} \qquad \text{III-1.17}$$

Introducing equations III-1.17 and III-1.15 into equation III-1.9, the following relationship between R(y) and $\varphi(y)$ is obtained:

$$R(y) = \frac{\varphi(y)}{\bar{u} - c} \frac{\partial \bar{\rho}}{\partial y} \qquad \text{III-1.18}$$

Thus,

$$\frac{\partial \rho'}{\partial x} = i\alpha \, R(y) \, e^{i\alpha(x-ct)} \qquad \text{III-1.19}$$

or

$$\frac{1}{i\alpha} \frac{\partial \rho'}{\partial x} e^{-i\alpha(x-ct)} = \frac{\varphi(y)}{\bar{u}-c} \frac{\partial \bar{\rho}}{\partial y} \qquad \text{III-1.20}$$

Substituting equation III-1.20 into equation III-1.16, the disturbance differential equation becomes

$$(\bar{u}-c)(\varphi''-\alpha^2\varphi) - \bar{u}''\varphi - g\frac{\bar{\rho}'}{\bar{\rho}} \frac{\varphi}{\bar{u}-c} + \frac{\bar{\rho}'}{\bar{\rho}}[(\bar{u}-c)\varphi-\bar{u}'\varphi] =$$

$$= \frac{\nu}{i\alpha}[\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi] \qquad \text{III-1.21}$$

Defining the reference length for velocity as δ and the reference length for density as $1/\beta$ and referring all velocities to a reference velocity U, equation III-1.21 may be put in the dimensionless form

$$(w-c)(\varphi''-\alpha^2\varphi)-w''\varphi-J\frac{\bar{\rho}'}{\bar{\rho}} \frac{\varphi}{w-c} + L\frac{\bar{\rho}'}{\bar{\rho}}[(w-c)\varphi-w'\varphi] =$$

$$= -\frac{i}{\alpha R}[\varphi^{IV} - 2\alpha^2\varphi'' + \alpha^4\varphi] \qquad \text{III-1.22}$$

In this equation, J is the Richardson number given by

$$J = \frac{g\beta \, \delta^2}{U^2} \qquad \text{III-1.23}$$

R is the Reynolds number given by

$$R = \frac{\delta U}{\nu} \qquad \text{III-1.24}$$

and L is the ratio of the scale length for velocity to that of density, i.e.,

$$L = \beta\,\delta \qquad \text{III-1.25}$$

Equation III-1.22 is the fundamental disturbance differential equation for density stratified flows neglecting the effect of molecular diffusion. This equation will be the starting point for the discussion in the remaining sections of this chapter.

III-2 Homogeneous Case

In this case, since there is no density variation, equation III-1.22 reduces to

$$(w-c)(\varphi''-\alpha^2\varphi) - w''\varphi = \frac{-i}{\alpha R}[\varphi^{IV} - 2\,\alpha^2\varphi'' + \alpha^4\varphi] \qquad \text{III-2.1}$$

This equation is commonly referred to as the "Orr-Sommerfeld equation". The solution technique to be discussed in this section was employed by Lessen[4] in his analysis of the stability of a homogeneous half-jet flow.

The independent variable in equation III-2.1 is y/δ. Lessen set the reference length δ equal to $\sqrt{\nu x/U}$. The independent variable of equation III-2.1 is then

$$y/\delta = y/\sqrt{\nu x/U} = y\sqrt{U/\nu x} = \eta \qquad \text{III-2.2}$$

In terms of the function $f(\eta)$, defined in reference 67 equation III-2.1 becomes

$$(f'-c)(\varphi''-\alpha^2\varphi - f'''\varphi) = \frac{-i}{\alpha R}[\varphi^{IV} - 2\alpha^2\varphi''+\alpha^4\varphi] \qquad \text{III-2.3}$$

Because equation III-2.3 is of the fourth order, a set of four linearly independent solutions exists. Thus,

$$\varphi = a\varphi_1 + b\varphi_2 + c\varphi_3 + d\varphi_4 \qquad \text{III-2.4}$$

Lessen showed that the solutions φ_3 and φ_4 are each unbounded somewhere in the flow field and therefore they cannot be considered in the solution of equation III-2.3. He solved for the solutions φ_1 and φ_2 by expanding φ in powers of $(-i/\alpha R)$. Thus,

$$\varphi = \sum_{k=0}^{\eta} \left(\frac{-i}{\alpha R}\right)^k \varphi^{(k)} \qquad \text{III-2.5}$$

Substituting this expression into equation III-2.3 and equating like powers of $(-i/\alpha R)$, the following equations are obtained for $\varphi^{(o)}$ and $\varphi^{(1)}$:

$$\varphi^{(o)''} - [\alpha^2 + \frac{f'''}{f'-c}]\, \varphi^{(o)} = 0 \qquad \text{III-2.6}$$

$$\varphi^{(1)''} - [\alpha^2 + \frac{f'''}{f'-c}]\, \varphi^{(1)} = \frac{\varphi^{(o)IV} - 2\alpha^2\varphi^{(o)''} + \alpha^4\varphi^{(o)}}{f'-c} \qquad \text{III-2.7}$$

An asymptotic expression for $f(\eta)$, valid for large negative values of η, is given by (reference 67),

$$f(\eta) = T_0 + T_1\, e^{1/2S\eta} + T_2\, e^{S\eta} + T_3\, e^{3/2S\eta} + \ldots$$

Since $f' < c$ for large negative η,

$$\frac{f'''}{f'-c} = D_1\, e^{1/2S\eta} + D_2\, e^{S\eta} + D_3\, e^{3/2S\eta} + \ldots \qquad \text{III-2.8}$$

The various coefficients in this expansion as well as the coefficients of other asymptotic expansions in this section are given in Table III-2.1.

Inserting equation III-2.8 into equation III-2.6,

$$\varphi^{(o)''} - [\alpha^2 + D_1 e^{1/2S\eta} + D_2 e^{S\eta} + D_3 e^{3/2S\eta}]\, \varphi^{(o)} = 0 \qquad \text{III-2.9}$$

From this equation, it follows that

$$\varphi^{(o)} = e^{\alpha\eta} + h_{1o}\, e^{(\alpha+1/2S)\eta} + h_{2o}\, e^{(\alpha+S)\eta} + h_{3o}\, e^{(\alpha+3/2S)\eta} + \ldots \qquad \text{III-2.10}$$

Inserting this expansion for $\varphi^{(o)}$ into equation III-2.7,

$$\varphi^{(1)''} - [\alpha^2 + D_1 e^{1/2S\eta} + D_2 e^{S\eta} + D_3 e^{3/2S\eta}]\, \varphi^{(1)} =$$

$$= P_1\, e^{(\alpha+1/2S)\eta} + P_2\, e^{(\alpha+S)\eta} + P_3\, e^{(\alpha+3/2S)\eta} \qquad \text{III-2.11}$$

Thus,

$$\varphi^{(1)} = e^{\alpha\eta} + h_{11}\, e^{(\alpha+1/2S)\eta} + h_{21}\, e^{(\alpha+S)\eta} + h_{31}\, e^{(\alpha+3/2S)\eta} + \ldots \qquad \text{III-2.12}$$

Lessen integrated equation III-2.6 by the method of analytic continuation. Because of the singularity in this equation at the point where $f' = c$, Lessen chose the path of numerical integration shown in Figure III-2.1.

It is first necessary to integrate the similarity differential equation $f''' + 1/2\, ff'' = 0$ along this path. To do this, the asymptotic expression for $f(\eta)$ given above is used to represent $f(\eta)$ for $\eta = -6-3i$. The method of analytic continuation

Table III-2.1.-Coefficients for Asymptotic Solutions of $\varphi^{(o)}$ and $\varphi^{(1)}$

Homogeneous Case

$$D_1 = \frac{-T_1 S^3}{8c}$$

$$D_2 = -\frac{S^3}{c}\left[T_2 + \frac{T_1{}^2 S}{16c}\right]$$

$$D_3 = -\frac{S^3}{c}\left[\frac{27T_3}{8} + \frac{5T_2 S}{8c} + \frac{T_1{}^3 S^2}{32c^2}\right]$$

$$E_o = -\frac{1}{c}$$

$$E_1 = -\frac{T_1 S}{2c^2}$$

$$E_2 = -\frac{S}{c^2}\left[T_2 + \frac{T_1{}^2 S}{4c}\right]$$

$$h_{1o} = \frac{D_1}{\alpha S + (\frac{S}{2})^2}$$

$$h_{2o} = \frac{D_1 h_{1o} + D_2}{2\alpha S + S^2}$$

$$h_{3o} = \frac{D_1 h_{2o} + D_2 h_{1o} + D_3}{3\alpha S + (\frac{3S}{2})^2}$$

$$G_1 = S^2(\alpha + 1/4S)^2 h_{1o}$$

$$G_2 = (2S)^2[\alpha + 1/2S]^2 h_{2o}$$

$$G_3 = (3S)^2[\alpha + 3/4S]^2 h_{3o}$$

$$PP_1 = E_o G_1$$

$$PP_2 = E_o G_2 + E_1 G_1$$

$$PP_3 = E_o G_3 + E_1 G_2 + E_2 G_1$$

$$h_{11} = \frac{D_1 + PP_1}{\alpha S + (\frac{S}{2})^2}$$

$$h_{31} = \frac{D_1 h_{21} + D_2 h_{11} + D_3 + PP_3}{3\alpha S + (\frac{3}{2}S)^2}$$

$$h_{21} = \frac{D_1 h_{11} + D_2 + PP_2}{2\alpha S + S^2}$$

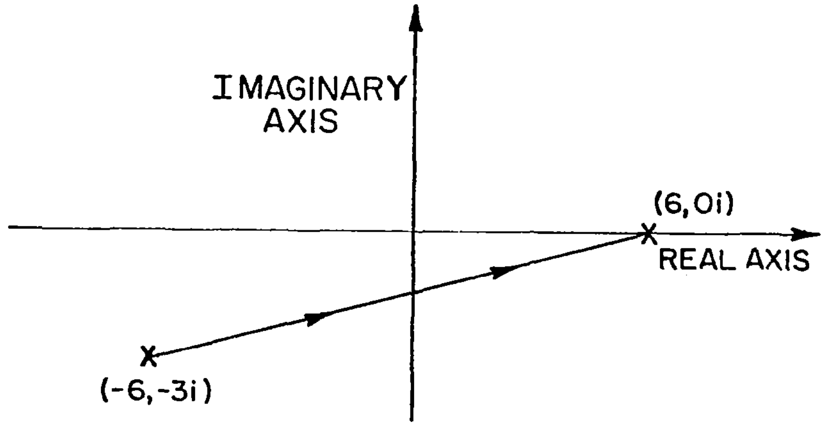

Fig. III -2.1 Path of Integration in Complex η Plane

is then used to integrate the asymptotic expression to $\eta = +6+0i$. The asymptotic expression for $\varphi^{(o)}$ given by equation III-2.10 is then used to represent $\varphi^{(o)}$ at $\eta = -6-3i$ and the method of analytic continuation is employed to integrate equation III-2.6 to $\eta = +6+0i$.

Finally, using equation III-2.12 to represent $\varphi^{(1)}$ for $\eta = -6-3i$, equation III-2.7 may be integrated to $\eta = +6+0i$. Lessen employed a finite difference scheme to perform this integration.

For large positive or negative values of η, equation III-2.3 may be written

$$\varphi'' - \alpha^2 \varphi = 0 \qquad \text{III-2.13}$$

The asymptotic form of φ is thus given by

$$\varphi(\pm\infty) = k_1 e^{\alpha\eta} + k_2 e^{-\alpha\eta} \qquad \text{III-2.14}$$

Because the solution must remain bounded, $k_1 = 0$, for $\eta \to \infty$ and $k_2 = 0$ for $\eta \to \infty$. It follows that the proper boundary conditions on φ are

$$\varphi' \to \alpha\varphi \qquad \eta \to -\infty$$

$$\varphi' \to -\alpha\varphi \qquad \eta \to \infty \qquad \text{III-2.15}$$

The function φ is approximated by

$$\varphi = \varphi^{(o)} + \left(\frac{-i}{\alpha R}\right) \varphi^{(1)} \qquad \text{III-2.16}$$

The boundary condition at plus infinity may be written

$$\varphi'(+\infty) + \alpha\,\varphi\,(+\infty) = 0 \qquad \text{III-2.17}$$

Substituting equation III-2.16 into equation III-2.17 and solving for the Reynolds number R gives

$$R = \frac{i}{\alpha}\left[\frac{\varphi^{(1)'}(+\infty) + \alpha\,\varphi^{(1)}(+\infty)}{\varphi^{(o)}(+\infty) + \alpha\,\varphi^{(o)}(-\infty)}\right] \qquad \text{III-2.18}$$

When $\varphi^{(o)}$ and $\varphi^{(1)}$ have been integrated to $\eta = +6+0i$, equation III-2.18 may be used to calculate the Reynolds number.

However, the coefficients of the asymptotic expressions for $\varphi^{(o)}$ and $\varphi^{(1)}$ contain the quantity c. Since c is unknown, the integration of equation III-2.3 becomes a trial and error process. The curve of neutral stability is desired. Therefore c_i is set equal to zero. For a given value of the wave number α, various values of c_r are assumed and equations III-2.6 and III-2.7 are integrated along the path shown in Figure III-2.1. Equation III-2.18 is then used to calculate the Reynolds number. The Reynolds number calculated in this way is usually complex. For a given value of α, that value of c_r for which the Reynolds number calculated from equation III-2.18 is real, is the desired value of c_r. This value of c_r along with the corresponding values of α and R form a set of eigenvalues. This process is repeated for various values of α to obtain various points on the curve of neutral stability in the α-R plane. Curves of equal amplification and damping may be similarly obtained by setting $c_i = 0$.

III -3 Heterogeneous Case

The disturbance differential equation in this case is

$$(w-c)(\varphi''-\alpha^2\varphi) - w''\varphi - J\frac{\bar{\rho}'}{\bar{\rho}}\frac{\varphi}{w-c} + L\frac{\bar{\rho}'}{\bar{\rho}}[(w-c)\varphi'-w'\varphi] =$$

$$= \frac{-i}{\alpha R}[\varphi^{IV} - 2\alpha^2\varphi'' , \alpha^4\varphi] \qquad \text{III-1.22}$$

Two additional terms are apparent in this equation as compared with equation III -2.1 which is for the homogeneous case. The term containing the factor L represents the effect of the heterogeneity of the fluid on the inertia and the term containing the factor J represents the effect of the gravity field (see section II -2).

The independent variable in equation III -1.22 is y/δ. If the reference length δ is again set equal to $\sqrt{\nu x/U}$, then the independent variable of equation III-1.22 is η, i.e.,

$$y/\delta = y/\sqrt{\nu x/U} = y\sqrt{U/\nu x} = \eta \qquad \text{III -2.2}$$

In terms of the functions $f(\eta)$ and $\rho(\eta)$ defined in section III -2, equation III -1.22 becomes

$$(f'c)(\varphi''-\alpha^2\varphi) - f'''\varphi - J \frac{Af''}{1-Af'} \frac{\varphi}{f'-C} +$$

$$+ L \frac{Af''}{1-Af'}[(f'-c)\varphi-f''\varphi] = \frac{-i}{\alpha R}[\varphi^{IV} -2\alpha^2\varphi'' + \alpha^4\varphi] \qquad \text{III-3.1}$$

Because equation III-3.1 is also of the fourth order, a set of four linearly independent solutions exists. However, Lessen's analysis for the homogeneous case may be extended to show that the φ_3 and $\dot{\varphi}_4$ solutions are again each unbounded somewh erein the flow field and therefore cannot be considered in the solution of equation III -3.1.

Lessen's solution method for the homogeneous case may be logically extended to this case. Thus, φ is again expanded in powers of $(-i/\alpha R)$, i.e.

$$\varphi = \sum_{k=0}^{\eta} (-i/\alpha R)^k \varphi^{(k)} \qquad \text{III -2.5}$$

Substituting this expression into equation III-3.1 and equating like powers of $(-i/\alpha R)$, the following equations are obtained for $\varphi^{(o)}$ and $\varphi^{(1)}$:

$$\varphi^{(o)''} - [\alpha^2 + \frac{f'''}{f'-c}] \varphi^{(o)} - J \frac{Af''}{1-Af'} \frac{\varphi^{(o)}}{(f'-c)^2} +$$

$$+ L \frac{Af''}{1-Af'} [\varphi^{(o)'} - \frac{f''}{f'-c} \varphi^{(o)}] = 0 \qquad \text{III-3.2}$$

$$\varphi^{(1)''} - [\alpha^2 + \frac{f'''}{f'-c}] \varphi^{(1)} - J \frac{Af''}{1-Af'} \frac{\varphi^{(1)}}{(f'-c)^2} +$$

$$+ L \frac{Af''}{1-Af'}[\varphi^{(1)} - \frac{f''}{f'-c} \varphi^{(1)}] = \frac{\varphi^{(o)} \quad -2\alpha^2\varphi^{(o)''} + \alpha^4\varphi^{(o)}}{f-c}$$

$$\text{III-3.3}$$

From reference 67, an asymptotic expression for $f(\eta)$ valid for large negative values of η, is given by

$$f(\eta) = T_0 + T_1\, e^{1/2S\eta} + T_2\, e^{S\eta} + T_3\, e^{3/2S\eta} + \dots$$

III-3.47

Since $f' < c$ for large negative η,

$$\frac{f'''}{f'-c} = D_1\, e^{1/2S\eta} + D_2\, e^{S\eta} + D_3\, e^{3/2S\eta} + \dots \qquad \text{III-3.4}$$

Similarly,

$$\frac{f''}{f-c} + R_1\, e^{1/2S\eta} + R_2\, e^{S\eta} + R_3\, e^{3/2S\eta} + \dots \qquad \text{III-3.5}$$

and

$$\frac{1}{(f'-c)^2} = AA_0 + AA_1\, e^{1/2S\eta} + AA_2\, e^{S\eta} + AA_3\, e^{3/2S\eta} + \dots$$

III-3.6

Since $Af' < 1$ for large negative η,

$$\frac{Af''}{1-Af'} = BB_1\, e^{1/2S\eta} + BB_2\, e^{S\eta} + BB_3\, e^{3/2S\eta} + \dots$$

III-3.7

Combining equations III-3.7 and III-3.6 gives

$$\frac{Af''}{1-Af'}\,\frac{1}{(f'-c)^2} = CC_1\, e^{1/2S\eta} + CC_2\, e^{S\eta} + CC_3\, e^{3/2S\eta} + \dots$$

III-3.8

The various coefficients in these expansions as well as the coefficients of other asymptotic expansions in this section are given in Table III-3.1

Inserting equations III-3.4, III-3.5 and III-3.8 into equation III-3.2,

Table III-3.1.-Coefficients for Asymptotic Solutions of $\varphi^{(o)}$ and $\varphi^{(1)}$

Heterogeneous Case
$D_1 = \frac{-T_1 S^3}{8c}$
$D_2 = -\frac{S^3}{c}\left[T_2 + \frac{T_1{}^2 S}{16c}\right]$
$D_3 = -\frac{S^3}{c}\left[\frac{27T_3}{8} + \frac{5T_2 S}{8c} + \frac{T_1{}^3 S^2}{32c^2}\right]$
$E_0 = -\frac{1}{c}$
$E_1 = -\frac{T_1 S}{2c^2}$
$E_2 = -\frac{S}{c^2}\left[T_2 + \frac{T_1{}^2 S}{4c}\right.$
$R_1 = -\frac{T_1 S^2}{4c}$
$R_2 = -\frac{S^2}{c}\left[T_2 + \frac{T_1{}^2 S}{8c}\right]$
$R_3 = -\frac{S^2}{c}\left[\frac{9T_3}{4} + \frac{3T_1 T_2 S}{4c} + \frac{T_1{}^3 S^2}{16c^2}\right]$
$BB_1 = \frac{AT_1 S^2}{4}$
$BB_2 = AT_2 S^2 + \frac{A^2 T_1{}^2 S^3}{8}$
$BB_3 = \frac{9AT_3 S^2}{4} + \frac{3A^2 T_1 T_2 S^3}{4} + \frac{A^3 T_1{}^2 S^4}{16}$

Table III-3.1. (Continued)

Heterogeneous Case
$DD_2 = BB_1R_1$
$DD_3 = BB_1R_2 + BB_2R_1$
$AA_0 = -\frac{1}{c^2}$
$AA_1 = -\frac{1}{c^3}\left(\frac{T_1S}{2}\right)$
$AA_2 = -\frac{1}{c^3}\left[T_2S + \frac{T_1^2S}{4c}\right]$
$CC_1 = AA_0BB_1$
$CC_2 = AA_1BB_1 + AA_0BB_2$
$CC_3 = AA_0BB_3 + AA_1BB_2 + AA_2BB_1$
$h_{10} = \dfrac{D_1 + JCC_1 - LBB_1\alpha}{\alpha s + (\frac{S}{2})^2}$
$FF_{20} = CC_1h_{10} + CC_2$
$EE_{20} = BB_1[\alpha + \frac{1}{2}S]h_{10}$
$h_{20} = \dfrac{D_1h_{10} + D_2 + JFF_{20} - L(EE_{20} + BB_2\alpha - DD_2)}{2\alpha S + S^2}$
$FF_{30} = CC_1h_{20} + CC_2h_{10} + CC_3$
$EE_{30} = BB_1(\alpha + S)h_{20} + BB_2(\alpha + \frac{1}{2}S)H_{10}$
$h_{30} = \dfrac{D_1h_{20} + D_2h_{10} + D_3 + JFF_{30} - L(EE_{30} + BB_3\alpha - DD_2h_{10} - DD_3)}{3\alpha S + (\frac{3S}{2})^2}$

Table III-3.1. (Continued)

Heterogeneous Case

$$G_1 = S^2 \ (\alpha + 1/4S)^2 \ h_{10}$$

$$G_2 = (2\ S)^2 \ [\alpha + 1/2S]^2 \ h_{20}$$

$$G_3 = (3S)^2 \ [\alpha + 3/4S]^2 \ h_{30}$$

$$PP_1 = E_0 G_1$$

$$PP_2 = E_0 G_2 + E_1 G_1$$

$$PP_3 = E_0 G_3 + E_1 G_2 + E_2 G_1$$

$$h_{11} = \frac{D_1 + JCC_1 - LBB_1\ \alpha + PP_1}{\alpha\ S + (S/2)^2}$$

$$EE_{21} = BB_1 (\alpha + 1/2S)\ h_{11}$$

$$FF_{21} = CC_1 h_{11} + CC_2$$

$$h_{21} = \frac{D_1 h_{11} + D_2 + JF_{21} - L(EE_{21} + BB_2\ \alpha - DD_2) + PP_2}{2\ \alpha\ S + S^2}$$

$$EE_{31} = BB_1 (\alpha + s)\ h_{21} + BB_2 (\alpha + 1/2S)\ h_{11}$$

$$FF_{31} = CC_1 h_{21} + CC_2 h_{11} + CC_3$$

$$h_{31} = \frac{D_1 h_{21} + D_2 h_{11} + D_3 + JFF_{31} - L(EE_{31} + BB_3 \alpha - DD_2 h_{11} - DD_3) + PP_3}{3\ \alpha\ S + (3S/2)^2}$$

$$\varphi^{(o)''} - [\alpha^2 + D_1\, e^{1/2S\eta} + D_2\, e^{S\eta} + D_3\, e^{3/2S\eta}]\, \varphi^{(o)}$$

$$- J[CC_1\, e^{1/2S\eta} + CC_2\, e^{S\eta} + CC_3\, e^{3/2S\eta}]\, \varphi^{(o)} +$$

$$+ L[BB_1\, e^{1/2S\eta} + BB_2\, e^{S\eta} + BB_3\, e^{3/2S\eta}][\varphi^{(o)'} - (R_1\, e^{1/2S\eta} + R_2\, e^{S\eta}$$

$$+ R_3\, e^{3/2S\eta})\varphi^{(o)}] = 0 \qquad \text{III-3.9}$$

or

$$\varphi^{(o)''} - [\alpha^2 + D_1\, e^{1/2S\eta} + D_2\, e^{S\eta} + D_3\, e^{3/2S\eta}]\, \varphi^{(o)}$$

$$-J\, [CC_1\, e^{1/2S\eta} + CC_2\, e^{S\eta},\ CC_3\, e^{3/2S\eta}]\, \varphi^{(o)}$$

$$+ L\, [BB_1 e^{1/2S\eta} + BB_2\, e^{S\eta} + BB_3\, e^{3/2S\eta}]\, \varphi^{(o)'}$$

$$- L\, [DD_2\, e^{S\eta} + DD_3\, e^{3/2S\eta}]\, \varphi^{(o)} = 0 \qquad \text{III-3.10}$$

From this equation, it follows that

$$\varphi^{(o)} = e^{\alpha\eta} + h_{1o}\, e^{(\alpha + \frac{1}{2}S)\eta} + h_{2o}\, e^{(\alpha+S)\eta} + h_{3o}\, e^{(\alpha+\frac{3}{2}S)\eta} + \ldots \qquad \text{III-3.11}$$

Inserting this expansion for $\varphi^{(o)}$ into equation III-3.3,

$$\varphi^{(1)''} - [\alpha^2 + D_1\, e^{1/2S\eta} + D_2\, e^{S\eta} + D_3\, e^{3/2S\eta}]\, \varphi^{(1)}$$

$$- J\, [CC_1\, e^{1/2S\eta} + CC_2\, e^{S\eta} + CC_3\, e^{3/2S\eta}]\, \varphi^{(1)}$$

$$+ L\, [BB_1\, e^{1/2S\eta} + BB_2\, e^{S\eta} + BB_3\, e^{3/2S\eta}]\, \varphi^{(1)'}$$

$$- L\, [DD_2\, e^{S\eta} + DD_3\, e^{3/2S\eta}]\, \varphi^{(1)}$$

$$= PP_1\, e^{(\alpha+\frac{1}{2}S)\eta} + PP_2\, e^{(\alpha+S)\eta} + PP_3\, e^{(\alpha+\frac{3}{2}S)\eta} \qquad \text{III-3.12}$$

Thus,

$$\varphi^{(1)} = e^{\alpha\eta} + h_{11}\, e^{(\alpha+\frac{1}{2}S)\eta} + h_{21}\, e^{(\alpha+S)\eta} + h_{31}\, e^{(\alpha+\frac{3}{2}S)\eta} + \dots$$

III-3.13

Using the asymptotic expressions for $\varphi^{(0)}$ and $\varphi^{(1)}$ given by equations III-3.11 and III-3.13 as a starting point, equations III-3.2 and III-3.3 may be integrated by the method of analytic continuation in a manner entirely analogous to that outlined in section III-2. The boundary conditions are given again by equations III-2.15. Thus, after integrating $\varphi^{(0)}$ and $\varphi^{(1)}$ to a large positive value of η, equation III-2.18 may again be used to calculate a Reynolds number. Three special cases may be cited.

Case 1 $J = 0$, $L \neq 0$

In this case, the value of L may be determined from the similarity solution of the boundary layer equations. For a given value of λ and Γ, L will be a constant. Once the value of L has been determined, the trial and error process to determine the curve of neutral stability would proceed as outlined for the homogeneous case in section III -2.

Case 2 $J \neq 0$, $L = 0$

In this case, J would be treated as a parameter. For a given value of λ and Γ, J would be specified and a neutral stability curve obtained to give a minimum critical

Reynolds number. The numerical value of J would then be changed and another Reynolds number obtained. In this way, the dependence of the minimum critical Reynolds number on the Richardson number could be explored.

Case 3 $J \neq 0$, $L \neq 0$

This case would be treated similarly to case 2. The only difference being that the value of L would be determined from the similarity solution of the boundary layer equations before introducing J as a parameter.

III-4 Stability Criteria

Stability criteria for viscous density stratified flows are not clearly established. The importance of the Richardson number and the Reynolds number is discussed in section II-2. It is generally accepted that the minimum critical Reynolds number increases as the Richardson number increases. However, stability criteria are generally given as a minimum critical Reynolds number for a given value of the Richardson number.

In section III-2 and III-3, the reference length for velocity δ was defined by

$$\delta = \sqrt{\nu x/U} \qquad \text{III-4.1}$$

The reference length for density will also be proportional to $\sqrt{\nu x/U}$. Thus,

$$1/\beta = (1/L)\sqrt{\nu x/U} \qquad \text{III-4.2}$$

$$\beta\delta = L \qquad \text{III-4.3}$$

The Richardson number J is given by

$$J = \frac{g\beta\delta^2}{U^2} = \frac{gL\delta}{U^2} \qquad \text{III-4.4}$$

and the Reynolds number is given by

$$R = \delta U/\nu \qquad \text{III-4.5}$$

It follows from equation III-4.1, that both the Richardson number and the Reynolds number increase with $\sqrt{x}$.

An increasing Reynolds number tends to be destabilizing whereas an increasing Richardson number tends to be stabilizing. What determines which of these factors is dominating?

A typical curve showing the increase in the minimum critical Reynolds number for increasing Richardson number is shown in Figure III-4.1.

Such a curve might be obtained for case 3 discussed in section III-3. The factor which must be determined is how the Richardson number and the Reynolds number change relative to one another as the flow proceeds downstream.

Equation III-4.5 may be rewritten

$$\delta = \frac{R\nu}{U} \qquad \text{III-4.6}$$

Substituting this into equation III-4.4 gives

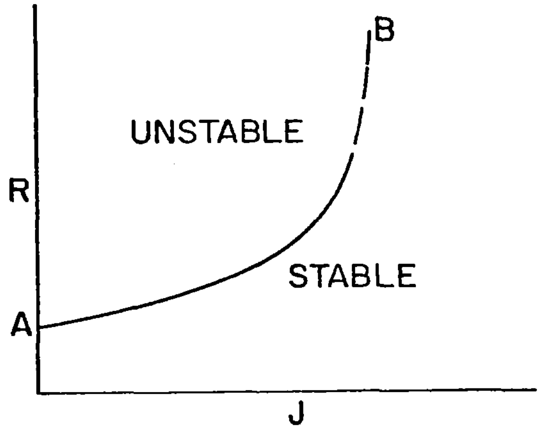

Fig. III-4.1. Critical Reynolds Number as a Function of Richardson Number

$$J = \frac{gI\delta}{U^2} = \frac{gIR\nu}{U^3} \qquad \text{III-4.7}$$

or

$$R = \frac{1}{I}\left(\frac{U^3}{g\,\nu}\right) J \qquad \text{III-4.8}$$

Equation III-4.8 indicates that the Reynolds number R is a linear function of the Richardson number J. The line representing the Reynolds number as a function of the Richardson number passes through the origin and has a slope of

$$\frac{1}{I}\left(\frac{U^3}{g\,\nu}\right) \qquad \text{III-4.9}$$

If the flow is to be stable, then the point on Figure III-4.1 representing the Reynolds number and the Richardson number for a certain downstream position must always lie below the curve A-B. The flow would be neutrally stable for that Reynolds number-Richardson number line which is tangent to the curve A-B. Thus, the slope of that particular line designates the critical value of the dimensionless group

$$\frac{1}{L}\left(\frac{U^3}{g\nu}\right) \qquad \text{III-4.9}$$

for stability. It can be easily shown that the dimensionless group $(U^3/g\nu)$ is the product of the Reynolds number and the Froude number.

IV-Results

The homogeneous stability problem for $\lambda = 1.0$ was considered by Lessen.[4] His solution is discussed in section II-1. In section III-3, an extension of Lessen's solution method to the heterogeneous stability problem was presented. In that section, three special cases were discussed. The case for which $J \neq 0$ and $L = 0$ is not particular applicable to the half-jet mixing problem since L is usually significantly different from zero. This case occurs in meterological applications where the density change takes place over a much wider region than the velocity change and therefore L is very small.

The case for which $J = 0$ and $L \neq 0$ corresponds to a heterogeneous half-jet type flow with no gravitational field. Neutral stability curves for this case are not available. It was proposed to calculate curves of neutral stability for flows with various density ratios. The solution method of Lessen is directly applicable only for this case of $\lambda = 1.0$, i.e., $u_2 = 0$. Analytical laminar similarity solutions were obtained for $\lambda = 1.0$ and $\Gamma = 0, -0.2, -0.4, -0.6$ and -0.8. The values of the constant S appearing in the asymptotic expressions for the solution of the boundary layer equations are given in Table IV-1.

Table IV-1. Values of the Constant S

Γ	S
-0.8	0.59063895
-0.6	0.74360338
-0.4	0.91674868
-0.2	1.07730919
0	1.23849623
0 (Lessen[a])	1.23849316

From the analytical similarity solutions, the value of L for each value of Γ was calculated. The width of the mixing region for density was defined in a manner entirely analogous to the way in which the width of mixing region for velocity was defined in reference 67. However, in this case the width of the mixing regions for both velocity and density was based on a 98% change across the mixing region rather than a 90% change as in section V-3. The results are shown in Table IV-2.

Table IV-2. - Values of the Constant L

Γ	L
-0.2	0.956
-0.4	0.901
-0.6	0.821
-0.75	0.727

It is interesting to note that the value of L decreases as the density ratio is increased. The numerical results of the similarity solution for these cases are given in reference 67 as well as values of α, β, η_0 and ξ_0.

The first neutral stability curve to be calculated was for the case of $\Gamma = -0.4$ corresponding to a density ratio of 7 to 3. Using the method outlined in section III-3, points on the curve of neutral stability were obtained by trial and error procedure. As the value of the wave number α was decreased, the eigenvalues of c_r and R decreased. However, for values of α less than 0.335, eigenvalues for c_r and real R could not be found. The results obtained are shown in Table IV-3.

Table IV-3. Eigenvalues of α, c_r and R

α	c_r	R	α	c_r	R
0.375	0.5324	238.23+1.03i	0.345	0.4800	35.71+0.83i
0.375	0.5325	240.93-1.03i	0.345	0.4835	37.46+0.02i
0.365	0.5175	84.32+0.92i	0.345	0.4850	38.25-0.40i
0.365	0.5179	85.58+0.06i	0.335	0.4580	27.87+0.10i
0.365	0.5181	86.22-0.40i	0.335	0.4600	28.42-0.07i
0.355	0.5015	51.89+0.32i	0.335	0.4630	29.28-0.37i
0.355	0.5020	52.43-0.01i	0.325	0.4000	18.45-0.74i
			0.325	0.4200	20.89-0.54i
			0.325	0.4350	23.18-0.76i

It can be seen from this table that for $\alpha = 0.325$, the imaginary part of R does not change sign as c_r is increased. Thus, a real eigenvalue of R for $\alpha = 0.325$ was not found. The reason for this characteristic is not known. A thorough search of the numerical procedure for algebraic or typographical errors was made. None were found. The eigenvalues interpolated for real R are given in Table IV-4.

Table IV-4. Eigenvalues of α, c_r and Real R

α	c_r	R
0.375	0.532	234.6
0.365	0.518	85.6
0.355	0.502	52.4
0.345	0.484	37.5
0.335	0.459	28.2

The computer program used to determine the eigenvalues in Table IV-4 was able to duplicate Lessen's eigenvalues for the homogeneous case exactly. The points on the neutral stability curve determined by Lessen for the homogeneous case are shown in Figure IV-1. In the same figure, the eigenvalues in Table IV-4 are shown. The cut off wave number for the flow with a 7 to 3 density ratio appears to be less than that for the homogeneous case. However, the

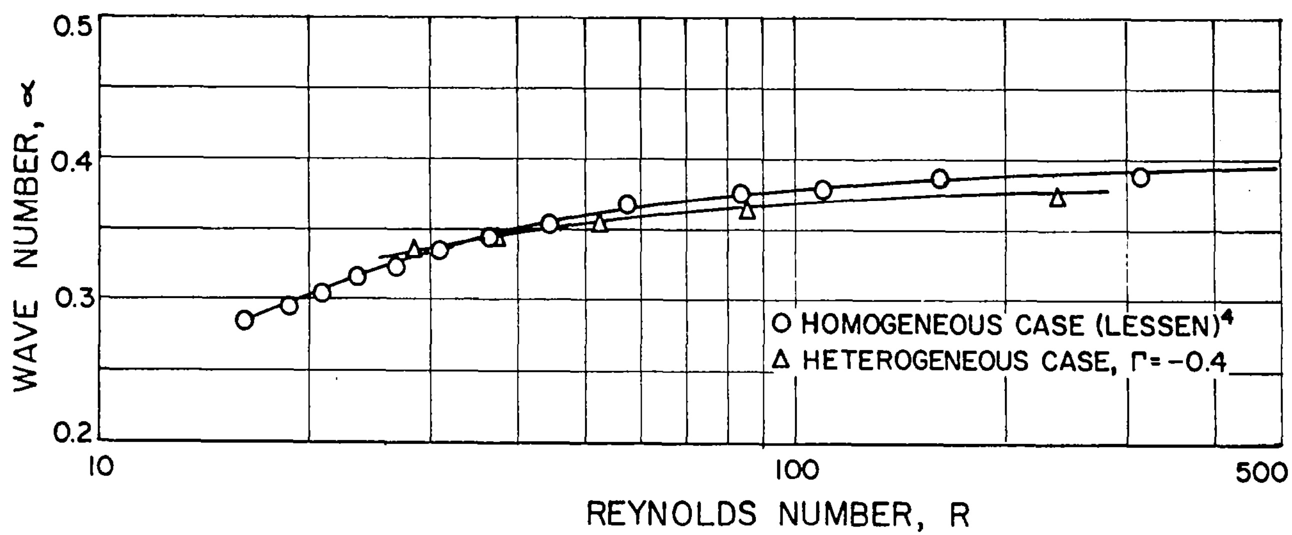

Fig. IV-1. Neutral Stability Curves for Half-Jet Type Flows two neutral stability curves cross at a value of the wave number of about 0.340.

Unfortunately, it was not possible to determine if the critical Reynolds number for the heterogeneous flow is higher than that for the homogeneous flow. If this was true, then the effect of the "heterogeneity of inertia" could definitely be said to be stabilizing. The fact that the value of the cut off wave number is less for the heterogeneous case is not a true indication of a stabilizing effect unless instability to a narrower band of wave-lengths can be said to be more stable than instability to a wider band of wave-lengths.

An attempt to calculate curves of equal amplification and damping led to very ambiguous results. Because of this and the inability to obtain real eigenvalues of R for a wide range of wave numbers, attempts to calculate neutral stability curves for other density ratios were not made. The potential results of these calculations if the reasons for the associated difficulties could be eliminated, should be very interesting and valuable. Results for the case $J \neq 0$ and $L \neq 0$ would also be of great interest and value.

BIBLIOGRAPHY

1. Tollmien, W., Berechnung turbulenter Ausbreitungsvorgange. ZAMM 6, 458-478 (1926) NACA TM 1085 (1945).

2. Rayleigh, Lord, Science Papers I, 474 (1880); III, 17 (1887), IV, 197 (1913).

3. Lessen, M., NACA Technical Note 929 (1950).

4. Goertler, H., Berchnung von Aufgaben der frein Turbulenz auf Grund eines neuen Naherungsansatzes. ZAMM 22, 244-253 (1942).

5. Kuethe, A., "Investigation of the Turbulent Mixing Regions Formed by Jets," J. Appl. Mech., 11, 3, A, 87, (1935).

6. Chiarulli, P., Wright-Patterson Air Force Base Headquarters Air Material Command Technical Report No. TS/1228-1A (1949).

7. Lin, C. C., NACA Technical Note 2887 (1953).

8. Szablewski, W., Zur Theorie der turbulenten Stromung von Gasen stark veranderlicher Dichte. Ing.-Arch. 20, 67 (1952).

9. Szablewski, W., Zeitliche Auflosung einer ebenen Trennungsflache der Geschwindigkeit und Dichte. ZAMM 35, 464-468 (1955).

10. Szablewski, W., Turbulente Vermischung zweier ebener Luftstrahlen von fast gleicher Geschwindigkeit und stark unterschiedlicher Temperature. Ing.-Arch. 20, 73 (1952).

11. Pai, S. I., "On Turbulent Jet Mixing of Two Gases at Constant Temperature," J. of Appl. Mech. 22, (1955).

12. Liepmann, H. W., and J. Laufer, NACA Tech. Note 1257 (1947).

13. Reichardt, H. Gesetzmassigkeiten der freien Turbulenz. VDI-Forschungsheft, 414 (1942), 2 ed. 1951.

14. Albertson, M. L., Dai, Y. B., Jensen, P. A., and Rouse, H., "Diffusion of Submerged Jets," Proc. Am. Soc. Civil engrs., 74, (1948) 1571.

15. Tollmien, W., Einallgemeines Kriterium der Instabilitat Laminarer Geschwindigkeitsverteilungen. Nachr. Gas. Wiss. Goettingen Math. Phys. Klasse, Fachgruppe I, 1, 79-114 (1935) Engl. transl. in NACA TM No. 792 (1936).

16. Lessen, M. and S. Ko, "Viscous Instability of an Incompressible Fluid Half-Jet Flow," Physics of Fluids, 9, 1179 (1966).

17. Heisenberg, W., Euber Stabilitaet und Turbulenz von Fluessigkeitsstroemen, Ann. Phys. 74, 577-627 (1924).

18. Lessen, M., Fox, J. A. and Zien, H., J. Fluid Mech. 23, 355 (1965).

19. Betchov, R. and Szewczyk, A., Phys. Fluids 6, 1391 (1963).

20. Curle, N., Aeronautical Research Council, Great Britain, Fluid Motion Sub-Committee Report F. M. 2400 (1956).

21. Esch, R., J. Fluid Mech. 3, 289 (1957).

22. Tatsumi, J., and Gotoh, K., J. Fluid Mech. 7, 433 (1960).

23. Carrier, G. F. and Chang, C. T., 1959 On An Initial Value Problem Concerning Taylor Instability of Incompressible Fluid. Quart. Appl. Math. 16, 4.

24. Menkes, J., "On The Stability of a Shear Layer", J. Fluid Mech. 5, 1959.

25. Richardson, L. F., "The Supply of Energy From and To Atmospheric Eddies." Proc. Roy. Soc. London, A 97, 354 (1926).

26. Prandtl, L., Einfluss Stabilisierender Krafte auf Die Turbulenze. Vortrage aus d. Geb. d. Aerodyn. u. Verwandter Gebiet, Aachen 1929, 1; Springer Berlin 1930.

27. Taylor, G. I., "Effects of Variation in Density on the Stability of Superposed Streams of Fluid." Proc. Roy. Soc. London, A132, 499 (1931).

28. Goldstein S., "On The Stability of Superposed Streams of Fluids of Different Densities." Proc. Roy. Soc. London, A 132, 523 (1939).

29. Drazin, P. G., "The Stability of a Shear Layer in an Unbounded Heterogeneous Inviscid Fluid." J. Fluid Mech., 4, 1958.

30. Miles, J. W., "The Stability of a Heterogeneous Shear Layer," J. Fluid Mech., 6, 583 (1959).

31. Case, K. M., "Stability of an Idealized Atmosphere. I. Discussion of Results," The Physics of Fluids, 3, 149-54 (1960).

32. Dikii, L. A., "On The Stability of Plane Parallel Flows of an Inhomogeneous Fluid," PPM 24, 249-257, 1960.

33. Chandrasekhar, S.,"Hydrodynamic and Hydromagnetic Stability,"Oxford Press, 1961.

34. Menkes, J., "On The Stability of a Heterogeneous Shear Layer Subject of a Body Force," J. Fluid Mech., 11, 1961.

35. Shen, S. F., "The Theory of Stability of Compressible Laminar Boundary Layers with Injection of a Foreign Gas", NAVORD Report 4467, 28, May 1957.

36. Schlichting, H., Turbulenz bei Warmeschichtung. ZAMM 15, 313 (1935), also Proc. Fourth Int. Congr. Appl. Mech., Cambridge, p. 245, 1935.

37. Prandtl, L., Bericht uber Untersuchungen zur Ausgebildeten Turbulenz. ZAMM 5, 136 (1925).

38. Taylor, G. I., "The Transport of Vorticity and Heat Through Fluids in Turbulent Motion," Proc. Roy. Soc., (London), A-135, 828, 685 (1932).

39. Reichardt, H., "Gesetzmassigkeiten der Freien Turbulenz," VDI-Forschungsh., 414 (1951).

40. Prandtl, L., "Bemerkung zur Theories der Freien Turbulenz," ZAMM 22, 5 (1942).

41. Abramovich, G. N., The Theory of Turbulent Jets, Massachusetts Institute of Technology Press, Cambridge, Mass., 1963.

42. Prandtl, L., The Mechanics of Viscous Fluids. In W. F. Durand, Aerodynamic Theory, III, 166 (1935). See also Proc. 2nd Intern. Congr. of Appl. Mech. Zurich 1926.

43. Yen, K. T., "On the Indeterminateness of the Boundary Conditions for the Mixing of Two Parallel Streams," Transactions of the ASME, p. 390, 1960.

44. Crane, L. J., "The Laminar and Turbulent Mixing of Jets of Compressible Fluid - Part II, The Mixing of Two Semi-Infinite Streams," Journal of Fluid Mechanics, 3, 1957.

45. Torda, T. P., Ackermann, W. O. and Burnett, H. R., "Symmetric Turbulent Mixing of Two Parallel Streams," J. Appl. Mech., 20, 1953, p. 63.

46. Chapman, A. J. and Korst, H. H., "Free Jet Boundary with Consideration of Initial Boundary Layer," Proc. 2nd U. S.Nat. Cong. of Appl. Mech., 1954, p. 723.

47. Ackerman, W. O., "Laminar Incompressible Mixing of Two Streams of Different Velocities," Tech. Report No. 14-TR-4, Contract No. AF 33(038)-21251, Aug. 1952.

48. Keulegan, G. H., "Laminar Flow at the Interface of Two Liquids," U. S. Bur. Stds., J. of Research, 32, (1944), p. 303.

49. Lock, R. S., "The Velocity Distribution in the Laminar Boundary Layer Between Parallel Streams," Quarterly Journal of Mech. and Appl. Math., 4, 1957, p. 81.

50. Potter, C. E., "Laminar Boundary Layers at the Interface of Co-Current Parallel Streams," Quart. J. of Mech. and Appl. Math., 10, 1957, p. 302.

51. Kobashi, Yasiyiro, "An Experiment on Turbulent Mixing," Proc. of 3rd Japan Nat. Cong. for Appl. Mech., 1953, p. 261.

52. Sato, H., "Experimental Investigation on the Transition of Laminar Separated Layer," J. of Phys. Soc. of Japan, 11, 1956.

53. Sato, H., "Further Investigation on the Transition of Two-Dimensional Separated Layer at Subsonic Speeds," J. of Phys. Soc. of Japan, 14, 1959.

54. Hauff, K., M. S. Thesis, Illinois Institute of Technology, 1966.

55. Ellison, T. H., and Turner, J. S., "Turbulent Entrainment in Stratified Flows," J. Fluid Mech. 6, 1959.

56. Minami, I., "On The Transport of Salinity in Stratified Turbulent Flows," Pro. 12 Japan Nat. Cong. for Appl. Mech, 1962, p. 133.

57. Webster, C. A. G., "An Experimental Study of Turbulence in a Density-Stratified Shear Flow," J. Fluid Mech. 19, 1964.

58. Okuda, S., "The Velocity Distribution at the Interface Between Two Liquids," J. Phys. Soc. of Japan 16, No. 2, 1961, p. 324.

59. Reichardt, H. and Prandtl, L., Einfluss von Warmeschichtung auf die Eigenschaften einer Turbulenten Stromung. Deutsche Forschung, No. 21, 110 (1934).

60. Schlichting, H., Boundary Layer Theory, Fourth Edition, McGraw-Hill Book Company, Inc., New York, 1960.

61. Hu, N., "On Turbulent Mixing of Two Fluids of Different Densities," Ph. D. Thesis, California Institute of Technology, 1944.

62. Yiç C., Quart. Appl. Math., 12, p. 434, 1955.

63. King, L. V., Phil. Trans. Roy. Soc. London, 214A, p. 373, (1914).

64. Van der Hegge Zijnen, B. G., Appl. Sci. Research, 6A, p. 129 (1956).

65. Kramers, H., Physica, 12, p. 61 (1946).

66. Hinze, J. O., Turbulence, McGraw-Hill Book Company, Inc. (1959).

67. Baker, R. L. and Weinstein, H., "Analytical Investigation of the Mixing of Two Parallel Streams of Dissimilar Fluids." NASA Contractor Report CR-956.

68. Baker, R. L. and Weinstein, H., "Experimental Investigation of the Mixing of Two Parallel Streams of Dissimilar Fluids." NASA Contractor Report CR-957.

NASA-Langley, 1968 —— 12 E-4128

National Aeronautics and Space Administration
WASHINGTON, D. C.

OFFICIAL BUSINESS

FIRST CLASS MAIL

POSTAGE AND FEES PAID
NATIONAL AERONAUTICS A
SPACE ADMINISTRATION

100 001 37 51 30S 68031 00903
AIR FORCE WEAPONS LABORATORY/AFWL/
KIRTLAND AIR FORCE BASE, NEW MEXICO 87117

ATT MISS MADELINE F. CANOVA, CHIEF TECHNIC

POSTMASTER: If Undeliverable (Section 1
Postal Manual) Do Not Retu

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Scientific and technical information generated under a NASA contract or grant and considered an important contribution to existing knowledge.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

TECHNOLOGY UTILIZATION PUBLICATIONS: Information on technology used by NASA that may be of particular interest in commercial and other non-aerospace applications. Publications include Tech Briefs, Technology Utilization Reports and Notes, and Technology Surveys.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546